EXISTENCE AND CONCENTRATION OF SOLUTIONS FOR A FRACTIONAL SCHRÖDINGER EQUATIONS WITH SUBLINEAR NONLINEARITY

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Abstract. This article concerns the fractional elliptic equations

$$(-\Delta)^s u + \lambda V(x) u = f(u), \quad u \in H^s(\mathbb{R}^N),$$

where $(-\Delta)^s$ ($s \in (0\,,1)$) denotes the fractional Laplacian, $\lambda > 0$ is a parameter, $V \in C(\mathbb{R}^N)$ and $V^{-1}(0)$ has nonempty interior. Under some mild assumptions, we establish the existence of nontrivial solutions. Moreover, the concentration of solutions is also explored on the set $V^{-1}(0)$ as $\lambda \to \infty$.

1. Introduction and statement of main results

We consider the nonlinear fractional Schrödinger equation

$$(-\Delta)^s u + \lambda V(x) u = f(u), \quad u \in H^s(\mathbb{R}^N), \tag{1.1}$$

where $(-\Delta)^s$ $(0 < s < 1)$ is the fractional Laplace operator, $\lambda > 0$ is a parameter, and $H^s(\mathbb{R}^N)$ is the usual fractional Sobolev space with the norm

$$\|u\|_{H^s} := \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2) dx \Big)^{\frac{1}{2}}.$$

The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. It was discovered by Nick Laskin as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like euantumn mechanical paths. Recently, a great attention has been devoted to the fractional and nonlocal operators of elliptic type, both for their interesting theoretical structure and in view of concrete applications in many fields. This type of operator has been studied by many authors [4, 5, 6, 7, 8, 11, 14] and references therein.

In [8], Felmer et al. proved the existence of positive solutions of nonlinear Schrödinger equation involving the fractional Laplacian in $\mathbb{R}^N$. For the whole space $\mathbb{R}^N$ case, the main difficulty of this problem is the lack of compactness for Sobolev embedding theorem. To overcome this difficulty, some authors assumed that the potential V satisfies some additional condition. Later, the authors in [11] considered the equation (1.1) with the critical exponent growth They proved that the energy functional possess the property of locally compact. In this paper, we are interested in the case that the nonlinearity f is sublinear and indefinite. To our knowledge, few works concerning on this case up to now. Motivated by the above articles, we continue to consider problem (1.1) with steep well potential and study

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the existence of nontrivial solution and concentration results under some mild assumptions different from those studied previously. To reduce our statements, we make the following assumptions for potential V:

(V_1) $V(x) \in C(\mathbb{R}^N)$ and $V(x) \geq 0$ on $\mathbb{R}^N$;

(V_2) There exists a constant $b > 0$ such that the set $V_b := \{x \in \mathbb{R}^N | V(x) < b\}$ is nonempty and has finite Lebesgue measure;

(V_3) $\Omega = \text{int}V^{-1}(0)$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

Based on the above assumptions, the main purpose of this paper is to prove the existence of nontrivial solutions and to investigate the concentration phenomenon of solutions on the set $V^{-1}(0)$ as $\lambda \to \infty$. This kind of potential λV satisfying $(V_1) - (V_3)$ is referred as the steep well potential. It is worth mentioning that some papers always assumed the potential $V(x) > 0$ for all $x \in \mathbb{R}^N$. Compared with the case $V > 0$, our assumptions on V are rather weak, and perhaps more important. To state our results, we need the following assumptions:

(f_1) $f \in C(\mathbb{R}^N, \mathbb{R})$ and there exist constants $1 < p < 2$ and functions $\xi(x) \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$ such that

$$|f(u)| \leq \xi(x)|u|^{p-1}, \quad \text{for all } u \in \mathbb{R}.$$

(f_2) There exist three constants $\eta, \delta > 0, \gamma \in (1, 2)$ such that

$$|F(u)| \geq \eta |u|^{\gamma} \quad \text{and all } x \in \Omega \text{ and } |u| \leq \delta,$$

where $F(u) = \int_0^u f(t)dt$.

On the existence of solutions we have the following result.

Theorem 1.1. *Assume that the conditions* $(V_1) - (V_3)$, (f_1) *and* (f_2) *hold. Then there exists* $\Lambda_0 > 0$ *such that for every* $\lambda > \Lambda_0$, *problem* (1.1) *has at least one solution* u_λ.

On the concentration of solutions we have the following result.

Theorem 1.2. *Let* u_λ *be a solution of problem* (1.1) *obtained in Theorem 1.1, then* $u_\lambda \to u_0$ *strongly in* $H^s(\mathbb{R}^N)$ *as* $\lambda \to \infty$, *where* u_0 *is a nontrivial solution of the equation*

$$\begin{cases} (-\Delta)^s u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

The paper is organized as follows. In Section 2, we give some preliminary results. In Section3, we finish the proof of Theorem 1.1. In Section 4, we study the concentration of solutions and prove Theorem 1.2.

2. Preliminary results

The fractional Laplacian $(-\Delta)^s$ with $s \in (0\,,1)$ of a function $u : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \forall s \in (0\,,1),$$

where $\mathcal{F}$ is the Fourier transform.

Recently, Caffarelli and Silvestre [4] developed a local interpretation of the fractional Laplacian given in $\mathbb{R}^N$ by considering a Neumann type operator in the extended domain $\mathbb{R}^{N+1}_+$ defined by $\{(x,t) \in \mathbb{R}^{N+1} : t > 0\}$. For $u \in H^s(\mathbb{R}^N)$, the

solution $w \in H^s_L(\mathbb{R}^{N+1}_+)$ of

$$\begin{cases} -div(t^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w = u & \text{in } \mathbb{R}^{N+} \times \{0\}, \end{cases}$$

is called s-harmonic extension $w = E_s(u)$ of u and it is proved in [4] that

$$\lim_{t\to 0^+} t^{1-2s}\frac{\partial w}{\partial t}(x,t) = -k_s(-\Delta)^s u(x),$$

where $k_s := 2^{1-2s}\Gamma(1-s)\Gamma(s)^{-1}$, the space $H^s_L(\mathbb{R}^{N+1}_+)$ is defined as the completion of $C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$ under the norm

$$\|w\|_{H^s_L} := \Big(k_s\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w(x,t)|^2 dxdt\Big)^{\frac{1}{2}},$$

A similar extension, for nonlocal problems on bounded domain Ω with the zero Dirichlet boundary condition was established. In this case, the space $H^s_{0,L}(\mathcal{C}_\Omega)$ is defined as the completion of $C_0^\infty(\overline{\mathcal{C}}_\Omega)$ under the norm

$$\|w\|_{H^s_{0,L}} := \Big(k_s\int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla w(x,t)|^2 dxdt\Big)^{\frac{1}{2}},$$

where $\mathcal{C}_\Omega := \Omega\times(0\,,+\infty)\subset\mathbb{R}^{N+1}_+$, some more detail see [2, 1].

In this paper, our problem (1.1) will be studied in the half-space, namely,

$$\begin{cases} -\,div(t^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\,k_s\dfrac{\partial w}{\partial \nu} = -\lambda V(x)w(x,0) + f(w(x,0)) & \text{in } \mathbb{R}^N\times\{0\}, \end{cases} \tag{2.1}$$

where

$$\frac{\partial w}{\partial \nu} := \lim_{t\to 0^+} t^{1-2s}\frac{\partial w}{\partial t}(x,t).$$

Consider the energy functional associated to (2.1) given by

$$\begin{aligned} J_\lambda(w) = {}& \frac{k_s}{2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w(x,t)|^2 dxdt + \frac{\lambda}{2}\int_{\mathbb{R}^N} V(x)|w(x,0)|^2 dx \\ & -\int_{\mathbb{R}^N} F(w(x,0))dx, \end{aligned} \tag{2.2}$$

which is C^1 with Cateaus derivative

$$\begin{aligned} \langle J'_\lambda(w), v\rangle = {}& k_s\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla w\cdot\nabla v dxdt + \lambda\int_{\mathbb{R}^N} V(x)w(x,0)v(x,0)dx \\ & -\int_{\mathbb{R}^N} f(w(x,0))v(x,0)dx, \end{aligned}$$

for all $w,\, v\in H^s_L(\mathbb{R}^{N+1}_+)$.

By the argument as above, if $w\in H^s_L(\mathbb{R}^{N+1}_+)$ is a critical point of J_λ, then $u = Tr(w)\in H^s(\mathbb{R}^N)$ is an energy or weak solution of problem (1.1). The converse is also right. By the equivalence of these two formulations, we will use both formulations in the sequel to their best advantage.

For $\lambda>0$. Let

$$E_\lambda = \Big\{w\in H^s_L(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + \lambda\int_{\mathbb{R}^N} V(x)|w(x,0)|^2 dx < +\infty\Big\},$$

be equipped with the norm

$$\|w\|_\lambda = \Big(k_s \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + \lambda \int_{\mathbb{R}^N} V(x)|w(x,0)|^2 dx\Big)^{1/2}.$$

It is clear that E_λ is a Hilbert space, and $\|w\|_1 \le \|w\|_\lambda$ for all $w \in E_1$ with $\lambda \ge 1$. Moreover, for all $w \in E_\lambda$, by using $(V_1)-(V_2)$ and Sobolev inequality, we have

$$\begin{aligned}
&k_s \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + \int_{\mathbb{R}^N} |w(x,0)|^2 dx\\
&= k_s \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + \int_{V_b} |w(x,0)|^2\, dx + \int_{\mathbb{R}^N\setminus V_b} |w(x,0)|^2\, dx\\
&\le k_s \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + |V_b|^{\frac{2^*-2}{2^*}}\Big(\int_{\mathbb{R}^N} |w(x,0)|^{2^*}\, dx\Big)^{2/2^*} + \int_{\mathbb{R}^N\setminus V_b} |w(x,0)|^2\, dx\\
&\le k_s \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + |V_b|^{\frac{2^*-2}{2^*}}\Big(\int_{\mathbb{R}^N} |w(x,0)|^{2^*}\, dx\Big)^{2/2^*}\\
&\quad + \frac{1}{\lambda b}\int_{\mathbb{R}^N\setminus V_b} \lambda V|w(x,0)|^2\, dx\\
&\le k_s \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + \frac{|V_b|^{\frac{2^*-2}{2^*}}}{S} k_s \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + \frac{1}{\lambda b}\int_{\mathbb{R}^N} \lambda V|w(x,0)|^2\, dx\\
&\le \max\Big\{1, 1+\frac{|V_b|^{\frac{2^*-2}{2^*}}}{S}, \frac{1}{\lambda b}\Big\} k_s \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + \int_{\mathbb{R}^N} \lambda V|w(x,0)|^2\, dx\\
&:= c_0 k_s \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2 dxdt + \int_{\mathbb{R}^N} \lambda V|w(x,0)|^2\, dx,
\end{aligned}$$

for

$$\lambda \ge \lambda_0 := \frac{S}{b(S+|V_b|^{\frac{2^*-2}{2^*}})}.$$

So, there exist positive constants λ_0 and c_0, independent of λ, such that

$$\|w\|_1 \le c_0\|w\|_\lambda, \quad \text{for all } u \in E_\lambda,\ \lambda \ge \lambda_0. \tag{2.3}$$

Furthermore, the embedding $E_\lambda \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for $p \in [2, 2_s^*]$, and $E_\lambda \hookrightarrow L^p_{Loc}(\mathbb{R}^N)$ is compact for $p \in [2, 2_s^*)$, i.e., there are constants $c_p > 0$ such that

$$\|w(x,0)\|_{L^p} \le c_p\|w\|_1 \le c_p c_0 \|w\|_\lambda, \quad \text{for all } u \in E_\lambda,\ \lambda \ge \lambda_0. \tag{2.4}$$

In order to prove Theorem 1.1, we use the following result by Rabinowitz [10].

Lemma 2.1. *Let E be a real Banach space and $\Phi \in C^1(E,\mathbb{R})$ satisfy the (PS)-condition. If Φ is bounded from below, then $c = \inf_E \Phi$ is a critical value of Φ.*

3. Proof of Theorem 1.1

In this section, we will finish the proof of Theorem 1.1. First, we give some useful lemmas.

Lemma 3.1. *Assume that $(V_1)-(V_3)$, (f_1) and (f_2) hold. Then there exists $\Lambda_0 > 0$ such that for every $\lambda > \Lambda_0$, J_λ is bounded from below in E_λ.*

Proof. From (2.4), (f_1) and the Hölder inequality, we have

$$\begin{aligned} J_\lambda(w) &= \frac{1}{2}\|w\|_\lambda^2 - \int_{\mathbb{R}^N} F(w(x,0))\,dx \\ &\geq \frac{1}{2}\|w\|_\lambda^2 - \Big(\int_{\mathbb{R}^N} |\xi(x)|^{\frac{2}{2-p}}\,dx\Big)^{(2-p)/2}\Big(\int_{\mathbb{R}^N} |w(x,0)|^2\,dx\Big)^{p/2} \\ &\geq \frac{1}{2}\|w\|_\lambda^2 - c_2^p c_0^p \|\xi\|_{L^{\frac{2}{2-p}}}\|w\|_\lambda^p, \end{aligned} \tag{3.1}$$

which implies that $J_\lambda(w) \to +\infty$ as $\|w\|_\lambda \to +\infty$, since $1 < p < 2$. Consequently, there exists $\Lambda_0 := \max\{1, \lambda_0\} > 0$ such that for every $\lambda > \Lambda_0$, J_λ is bounded from below and coerciveness on E_λ. □

Lemma 3.2. *Suppose that* $(V_1)-(V_3)$, (f_1) *and* (f_2) *are satisfied. Then* J_λ *satisfies the (PS)-condition for each* $\lambda > \Lambda_0$.

Proof. Assume that $\{w_n\} \subset E_\lambda$ is a sequence such that $J_\lambda(w_n)$ is bounded and $J'_\lambda(w_n) \to 0$ as $n \to \infty$. By Lemma 3.1, it is clear that $\{w_n\}$ is bounded in E_λ. Thus, there exists a constant $C > 0$ such that

$$\|w_n(x,0)\|_{L^p} \leq c_p c_0 \|w_n\|_\lambda \leq C, \quad \text{for all } w \in E_\lambda,\ \lambda \geq \lambda_0, \tag{3.2}$$

where $2 \leq p \leq 2_s^*$. Passing to a subsequence if necessary, we may assume that $w_n \rightharpoonup w_0$ weakly in E_λ. For any $\epsilon > 0$, since $\xi(x) \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, we can choose $R_\epsilon > 0$ such that

$$\Big(\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |\xi(x)|^{\frac{2}{2-p}}\,dx\Big)^{(2-p)/2} < \epsilon. \tag{3.3}$$

From $E_\lambda \hookrightarrow L^p$ and $w_n \rightharpoonup_0$ weakly in E_λ, we have $w_n(x,0) \to w_0(x,0)$ strongly in $L^2_{Loc}(\mathbb{R}^N)$. Hence

$$\lim_{n\to\infty} \int_{B_{R_\epsilon}} |w_n(x,0) - w_0(x,0)|^2\,dx = 0. \tag{3.4}$$

Therefore, from (3.4), there exists $N_0 \in \mathbb{N}$ such that

$$\int_{B_{R_\epsilon}} |w_n(x,0) - w_0(x,0)|^2\,dx < \epsilon^2, \quad \text{for } n \geq N_0. \tag{3.5}$$

Hence, by (f_1), (3.2), (3.5) and the Hölder inequality, for any $n \geq N_0$, we have

$$\begin{aligned} &\int_{B_{R_\epsilon}} |f(w_n(x,0)) - f(w_0(x,0))|\,|w_n(x,0) - w_0(x,0)|\,dx \\ &\leq \Big(\int_{B_{R_\epsilon}} |f(w_n(x,0)) - f(w_0(x,0))|^2\,dx\Big)^{1/2}\Big(\int_{B_{R_\epsilon}} |w_n(x,0) - w_0(x,0)|^2\,dx\Big)^{1/2} \\ &\leq \epsilon\Big(\int_{B_{R_\epsilon}} 2\left(|f(w_n(x,0))|^2 + |f(w_0(x,0))|^2\right)\,dx\Big)^{1/2} \\ &\leq 2\epsilon\Big[\Big(\int_{B_{R_\epsilon}} |\xi(x)|^2\left(|w_n(x,0)|^{2(p-1)} + |w_0(x,0)|^{2(p-1)}\right)\,dx\Big)^{1/2}\Big] \\ &\leq 2\epsilon\Big[\|\xi\|^2_{L^{\frac{2}{2-p}}}\left(\|w_n(x,0)\|_2^{2(p-1)} + \|w_0(x,0)\|_{L^2}^{2(p-1)}\right)\Big]^{1/2} \\ &\leq 2\epsilon\Big[\|\xi\|^2_{L^{\frac{2}{2-p}}}\left(C^{2(p-1)} + \|w_0(x,0)\|_{L^2}^{2(p-1)}\right)\Big]^{1/2}. \end{aligned} \tag{3.6}$$

On the other hand, by (3.2), (3.3), (3.5) and (f_1), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |f(w_n(x,0)) - f(w_0(x,0))|\,|w_n(x,0) - w_0(x,0)|\,dx \\
&\leq 2\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |\xi(x)|\left(|w_n(x,0)|^p + |w_0(x,0)|^p\right)\,dx \\
&\leq 2\epsilon c_2^p c_0^p\left(\|w_n\|_\lambda^p + \|w_0\|_\lambda^p\right) \\
&\leq 2\epsilon c_2^p c_0^{\gamma_i}\left(C^p + \|w_0\|_\lambda^p\right).
\end{aligned} \tag{3.7}$$

Since ϵ is arbitrary, combining (3.6) with (3.7), we have

$$\int_{\mathbb{R}^N} |f(w_n(x,0)) - f(w_0(x,0))|\,|w_n(x,0) - w_0(x,0)|\,dx < \epsilon, \tag{3.8}$$

as $n \to \infty$. Hence,

$$\begin{aligned}
&\langle J_\lambda'(w_n) - J_\lambda'(w_0), w_n - w_0\rangle = \|w_n - w_0\|_\lambda^2 \\
&+ \int_{\mathbb{R}^N}(f(w_n(x,0)) - f(w_0(x,0)))(w_n(x,0) - w_0(x,0))\,dx.
\end{aligned} \tag{3.9}$$

From, $\langle J_\lambda'(w_n) - J_\lambda'(u_0), w_n - w_0\rangle \to 0$, (3.8) and (3.9), we get $w_n \to w_0$ strongly in E_λ. Hence, J_λ satisfies (PS)-condition. □

Proof of Theorem 1.1. From Lemmas 2.1, 3.1, 3.2, we know that $c_\lambda = \inf_{E_\lambda} J_\lambda(w)$ is a critical value of functional J_λ; that is, there exists a critical point $w_\lambda \in E_\lambda$ such that $J_\lambda(w_\lambda) = c_\lambda$. Next, similar to the argument in [12], we show that $w_\lambda \neq 0$. Let $w^* \in H^s_{0,L}(\mathcal{C}_\Omega) \setminus \{0\}$ and $\|w^*\|_{L^\infty} \leq 1$, then by (f_2), we have

$$\begin{aligned}
J_\lambda(tw^*) &= \frac{1}{2}\|tw^*\|_\lambda^2 - \int_{\mathbb{R}^N} F(tw^*(x,0))\,dx \\
&= \frac{t^2}{2}\|w^*\|_\lambda^2 - \int_\Omega F(tw^*(x,0))\,dx \\
&\leq \frac{t^2}{2}\|w^*\|_\lambda^2 - \eta t^\gamma \int_\Omega |w^*|^\gamma\,dx,
\end{aligned} \tag{3.10}$$

where $0 < t < \delta$, δ be given in (f_2). Since $1 < \gamma < 2$, it follows from (3.10) that $J_\lambda(tw^*) < 0$ for $t > 0$ small enough. Hence, $J_\lambda(w_\lambda) = c_\lambda < 0$, therefore, w_λ is a nontrivial critical point of J_λ and so w_λ is a nontrivial solution of problem (2.1), that is, $u_\lambda(x) := Tr(w_\lambda) = w_\lambda(x,0)$ is a nontrivial solution of problem (1.1). The proof is complete. □

4. Concentration of solutions

In the following, we study the concentration of solutions for problem (1.1) as $\lambda \to \infty$. Define

$$\tilde{c} = \inf_{w \in H^s_{0,L}(\mathcal{C}_\Omega)} J_\lambda|_{H^s_{0,L}(\mathcal{C}_\Omega)}(w),$$

where $J_\lambda|_{H^s_{0,L}(\mathcal{C}_\Omega)}$ is a restriction of J on $H^s_{0,L}(\mathcal{C}_\Omega)$; that is,

$$J_\lambda|_{H^s_{0,L}(\mathcal{C}_\Omega)}(w) = \frac{k_s}{2}\int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla w|^2 dxdt - \int_\Omega F(w(x,0))dx,$$

for $w \in H_L^s(\mathbb{R}_+^{N+1})$. Similar to the proof of Theorem 1.1, it is easy to prove that $\tilde{c} < 0$ can be achieved. Since $H_{0,L}^s(\mathcal{C}_\Omega) \subset E_\lambda$ for all $\lambda > 0$, we get

$$c_\lambda \le \tilde{c} < 0, \quad \text{for all } \lambda > \Lambda_0.$$

Proof of Theorem 1.2. We follow the arguments in [3]. For any sequence $\lambda_n \to \infty$, let $w_n := w_{\lambda_n}$ be the critical points of J_{λ_n} obtained in Theorem 1.1. Thus

$$J_{\lambda_n}(w_n) \le \tilde{c} < 0 \tag{4.1}$$

and

$$\begin{aligned} J_{\lambda_n}(w_n) &= \frac{1}{2}\|w_n\|_{\lambda_n}^2 - \int_{\mathbb{R}^N} F(w_n(x,0))dx \\ &\ge \frac{1}{2}\|w_n\|_{\lambda_n}^2 - c_2^p c_0^p \|\xi\|_{\frac{2}{2-p}} \|w_n\|_{\lambda_n}^p, \end{aligned}$$

which implies

$$\|w_n\|_{\lambda_n} \le C, \tag{4.2}$$

where the constant $C > 0$ is independent of λ_n. Therefore, we may assume that $w_n \rightharpoonup w_0$ in E_λ and $w_n(x,0) \to w_0(x,0)$ in $L_{\text{loc}}^p(\mathbb{R}^N)$ for $2 \le p < 2_s^*$. From Fatou's lemma, we have

$$\int_{\mathbb{R}^N} V(x)|w_0(x,0)|^2\,dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)|w_n(x,0)|^2\,dx \le \liminf_{n\to\infty} \frac{\|w_n\|_{\lambda_n}^2}{\lambda_n} = 0,$$

which implies that $w_0 = 0$ a.e. in $\mathbb{R}^N \setminus \overline{V^{-1}(0)}$ and $w_0 \in H_{0,L}^s(\mathcal{C}_\Omega)$ by (V_3). Now for any $\varphi \in C_0^\infty(\mathcal{C}_\Omega)$, since $\langle J'_{\lambda_n}(w_n), \varphi\rangle = 0$, it is easy to verify that

$$k_s \int_{\mathcal{C}_\Omega} t^{1-2s} \nabla w_0 \cdot \nabla\varphi dxdt - \int_\Omega f(w_0(x,0))\varphi\,dx = 0,$$

which implies that $u_0(x) := Tr(w_0)$ is a weak solution of equation (1.2) by the density of $C_0^\infty(\overline{\mathcal{C}}_\Omega)$ in $H_{0,L}^s(\mathcal{C}_\Omega)$.

Next, we show that $w_n(x,0) \to w_0(x,0)$ in $L^p(\mathbb{R}^N)$ for $2 \le p < 2_*$. Otherwise, by Lions vanishing lemma [9, 13], there exist $\delta > 0, \rho > 0$ and $(x_n, y) \in \mathbb{R}_+^{N+1}$ such that

$$\int_{B_\rho^+ \cap \{y=0\}} |w_n - w_0|^2\,dx \ge \delta,$$

where $B_\rho^+ := \{(x,y) : |(x,y) - (x_n,y)| < \rho, y > 0\}$, and its base denotes by B_ρ. Since $w_n(x,0) \to w_0(x,0)$ in $L_{Loc}^2(\mathbb{R}^N)$, let $|x_n| \to \infty$, we have $|B_\rho \cap V_b| \to 0$. By the Hölder inequality, we get

$$\int_{B_\rho \cap V_b} |w_n(x,0) - w_0(x,0)|^2\,dx \le |B_\rho \cap V_b|^{\frac{2_*-2}{2_*}} \Big(\int_{\mathbb{R}^N} |u_n - u_0|^{2_*}\Big)^{2/2_*} \to 0.$$

Consequently,

$$\begin{aligned} \|w_n\|_{\lambda_n}^2 &\ge \lambda_n b \int_{B_\rho \cap V_b^\perp} |w_n(x,0)|^2 dx \\ &= \lambda_n b \Big(\int_{B_\rho \cap V_b^\perp} |w_n(x,0) - w_0(x,0)|^2\,dx + \int_{B_\rho \cap V_b^\perp} |w_0(x,0)|^2 dx\Big) + o(1) \\ &\ge \lambda_n b \Big(\int_{B_\rho} |w_n(x,0) - w_0(x,0)|^2\,dx - \int_{B_\rho \cap V_b} |w_n(x,0) - w_0(x,0)|^2\,dx\Big) + o(1) \\ &\to \infty \text{ as } n \to \infty, \end{aligned}$$

which contradicts (4.2). By virtue of $\langle J'_{\lambda_n}(w_n), w_n\rangle = \langle J'_{\lambda_n}(w_n), w_0\rangle = 0$ and the fact that $w_n(x,0) \to w_0(x,0)$ strongly in $L^p(\mathbb{R}^N)$ for $2 \le p < 2_s^*$, we have

$$\lim_{n\to\infty} \|w_n\|^2_{\lambda_n} = \|w_0\|^2_{\lambda_n}.$$

Hence, $w_n \to w_0$ strongly in E_λ. Moreover, from (4.1), we have $w_0 \neq 0$. This completes the proof. □

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